Paradoxes and Inclosure

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Readings

Suggested:

- Priest, G. (1994). The structure of the paradoxes of self-reference. *Mind*, 103(409), 25–34.
- Priest, G. (2010). Inclosures, vagueness, and self-reference.

Further:

- Bolander, T. (2024). Self-reference and paradox.
- Yablo, S. (1993). Paradox without self-reference.
- Yanofsky, N. S. (2003). A universal approach to self-referential paradoxes, incompleteness and fixed points. *Bulletin of Symbolic Logic*, 9(3), 362–386.
- Abramsky, S., & Zvesper, J. (2015). From Lawvere to Brandenburger-Keisler.
- *The logic of quantum paradoxes*, Samson Abramsky: `https://www.youtube.com/watch?v=_wGu7ra0lHY`

Outline

What counts as a self-reference paradox?

A working characterization:

- A system $\mathcal{S}$ (language, theory, concept) contains resources to
  1. encode or represent its own semantic or structural features, and
  2. apply some operation to the totality of objects bearers of such features,
  3. generating a contrast between *closure* inside the system and *transcendence* outside the system.
- The resulting argument yields an *inconsistent* or *limitative* outcome.

The Liar

Let λ be the sentence:

$$\lambda \equiv \neg \mathsf{T}('\lambda')$$

- If T obeys the (naive) T-schema:

$$\mathsf{T}('\varphi') \leftrightarrow \varphi$$

  then:

$$\lambda \leftrightarrow \neg\lambda$$

- Hence λ is both true and not true (classically: contradiction).

Russell's paradox

In naive comprehension:

$$R = \{x : x \notin x\}$$

Then:

$$R \in R \;\leftrightarrow\; R \notin R$$

- The contradiction does not require semantic vocabulary.
- It is often read as motivating restriction of comprehension.

Burali-Forti

Let On be “the set of all ordinals”. Define $\delta(X)$ as the least ordinal strictly greater than every member of X.

- Then $\delta(\mathrm{On})$ is an ordinal greater than all ordinals.
- So $\delta(\mathrm{On}) \in \mathrm{On}$ and $\delta(\mathrm{On}) \notin \mathrm{On}$.

Grelling-Nelson (Heterological)

Call a predicate *autological* iff it applies to itself, *heterological* otherwise. Let $H(x)$ mean "x is heterological."

Now we ask: is H heterological?

$$H(H) \leftrightarrow \neg H(H)$$

Ramsey's two families

Ramsey (1925) distinguishes:

- **Group A:** "purely logical/mathematical" paradoxes (e.g., Russell, Burali-Forti),
- **Group B:** "language/meaning" paradoxes (e.g., Liar, heterological).

But:

- Semantic and syntactic notions can be coded arithmetically or set-theoretically.
- The vocabulary boundary between "mathematics" and "metalanguage" shifts.

Priest argues that a **structural criterion** is preferable to a vocabulary-based one.

Outline

From Russell to Inclosure

Priest's diagnosis: many self-referential paradoxes arise from a tension between

- **Closure:** the relevant construction stays *inside* a totality,
- **Transcendence:** the same construction *escapes* any admissible sub-totality.

Russell's Schema

Let φ be a predicate and assume:

1. **Existence:** $\Omega = \{x : \varphi(x)\}$ exists.
2. **Transcendence:** For all $X \subseteq \Omega$, $\delta(X) \notin X$.
3. **Closure:** For all $X \subseteq \Omega$, $\delta(X) \in \Omega$.

Then, for $X = \Omega$:

$$\delta(\Omega) \in \Omega \text{ and } \delta(\Omega) \notin \Omega$$

$$\varphi(x) \equiv x \notin x$$

$$\Omega = \{x : x \notin x\} \quad \text{(Russell's set).}$$

$$\delta(X) = X$$

The Inclosure Schema

The generalization adds a *qualification* θ:

1. $\Omega = \{x : \varphi(x)\}$ and $\theta(\Omega)$.
2. If $X \subseteq \Omega$ and $\theta(X)$ then:
   - **Transcendence:** $\delta(X) \notin X$.
   - **Closure:** $\delta(X) \in \Omega$.

Thus $X = \Omega$ yields the inclosure contradiction:

$$\delta(\Omega) \notin \Omega \text{ and } \delta(\Omega) \in \Omega$$

We can use this to unify semantic, set-theoretic, definability, and even vagueness-based paradoxes.

Why the qualification θ matters

Many paradoxes require a restriction on admissible subcollections:

- "definable subsets",
- "nameable sets of sentences",
- "tolerant steps in a sorites sequence",
- "epistemically accessible states" (in multi-agent variants).

Set theory (as inclosures)

In these classical set-theoretic cases, the admissibility condition is trivial:

$$\theta(X) \equiv \top \quad \text{for all } X \subseteq \Omega$$

That is, *every* subcollection of Ω is eligible for the schema.

- **Russell:**
  $\varphi(x) \equiv x \notin x \qquad \Omega = \{x : x \notin x\} \qquad \theta(X) \equiv \top$
  $\delta(X) = X$
  Then at $X = \Omega$:
  $$\delta(\Omega) = \Omega \in \Omega \ \text{ and } \ \delta(\Omega) = \Omega \notin \Omega$$
- **Burali–Forti:**
  $\varphi(x) \equiv x$ is an ordinal $\qquad \Omega = \mathrm{On} \qquad \theta(X) \equiv \top$
  $\delta(X) =$
  the least ordinal strictly greater than every member of X.
  Then at $X = \Omega$:
  $$\delta(\mathrm{On}) \in \mathrm{On} \ \text{ and } \ \delta(\mathrm{On}) \notin \mathrm{On}$$

Cantor's theorem (as diagonal argument)

There is no surjection $f : X \to \mathcal{P}(X)$.
Proof: Assume for reductio that f is onto. We define the anti-diagonal set

$$C = \{x \in X : x \notin f(x)\}$$

Since f is surjective, $\exists c \in X$ such that $f(c) = C$. Then

$$c \in C \leftrightarrow c \notin f(c) \leftrightarrow c \notin C$$

A contradiction. Hence no surjection $X \to \mathcal{P}(X)$ exists and

$$|X| < |\mathcal{P}(X)|$$

	x_1	x_2	x_3
x_1	✓	✗	✓
x_2	✓	✗	✓
x_3	✗	✓	✓
C	✗	✓	✗

Cell (i, j) indicates $x_j \in f(x_i)$. Row C flips the diagonal condition $x_i \in f(x_i)$.

Cantor as an Inclosure

Fix a set X. Let $\Omega = \mathcal{P}(X)$ $\quad\varphi(u) \equiv u \subseteq X$

$$\theta(S) \equiv \exists f : X \to \Omega \; (\mathrm{ran}(f) = S)$$

So the admissible $S \subseteq \Omega$ are exactly those *representable as the range of some listing* f.

Diagonal/escape operator. Given $\theta(S)$, choose a witness f with $\mathrm{ran}(f) = S$ and define

$$\delta(S) = \{x \in X : x \notin f(x)\}$$

Transcendence. If $\theta(S)$ then $\delta(S) \notin S$. (Otherwise $\delta(S) = f(c)$ for some c, and $c \in \delta(S) \leftrightarrow c \notin \delta(S)$.)

Closure. For any $S \subseteq \Omega$ with $\theta(S)$, $\delta(S) \subseteq X$. Hence $\delta(S) \in \mathcal{P}(X) = \Omega$.

If we additionally assumed $\mathrm{ran}(f) = \Omega$ (i.e. f is surjective), then with $S = \Omega$: $\delta(\Omega) \notin \Omega$ and $\delta(\Omega) \in \Omega$. Thus no surjection $X \to \mathcal{P}(X)$ exists.

The Liar via inclosure

Let L be a language containing:

- a unary truth predicate $T(x)$ for codes of L-sentences, and
- a device for *naming* certain sets of sentences.

$$\Omega \; = \; \{\varphi \in L : T('\varphi')\}$$

$$\theta(X) \; \equiv \; \text{"}X \subseteq \Omega \text{ and there is a name } N_X \text{ in } L \text{ that denotes } X.\text{"}$$

Given such N_X, let $\delta(X)$ be a sentence λ_X satisfying the condition:

$$\lambda_X \; \leftrightarrow \; \neg('\lambda'_X \in N_X)$$

Transcendence. Assume $\theta(X)$. If $\lambda_X \in X$, then $\lambda_X \in \Omega$, hence λ_X is true, so $\neg('\lambda'_X \in N_X)$ But N_X names X, so $'\lambda'_X \notin N_X$ iff $\lambda_X \notin X$. Thus $\lambda_X \notin X$.

Closure. From $\lambda_X \notin X$ and the correctness of the name N_X, we get $\neg('\lambda'_X \in N_X)$, hence λ_X is true, so $\lambda_X \in \Omega$.

Therefore, at $X = \Omega$: $\delta(\Omega) \notin \Omega$ and $\delta(\Omega) \in \Omega$

Sorites

We model a sorites series as a finite *ordered* sequence

$$A = \langle a_0, a_1, \ldots, a_n \rangle$$

Successive items are "imperceptibly different". Let P be a vague predicate with $P(a_0)$ and $\neg P(a_n)$.

Tolerance: For each $i < n$: $P(a_i) \;\rightarrow\; P(a_{i+1})$

Totality: $\Omega = \{a_i \in A : P(a_i)\}$

Admissibility. Here θ encodes the "cut" assumption appropriate to sorites:

$$\theta(X) \;\equiv\; \exists k \leq n \; (X = \{a_i : i < k\})$$

So the admissible X are exactly the (possibly empty) *initial segments* of A.

For $\theta(X)$, define $\delta(X)$ as the *first element of A not in X*:

$$\delta(X) = a_k \quad \text{where } k = \min\{i \leq n : a_i \notin X\}$$

Sorites

Transcendence. If $\theta(X)$, then by definition: $\delta(X) \notin X$.

Closure. If $X \subseteq \Omega$ and $\theta(X)$, then:

- if $k = 0$, $\delta(X) = a_0 \in \Omega$ since $P(a_0)$;
- if $k > 0$, then $a_{k-1} \in X \subseteq \Omega$, so $P(a_{k-1})$, hence by tolerance $P(a_k)$, i.e. $\delta(X) \in \Omega$.

Assuming $\theta(\Omega)$ (i.e. the P-items form an admissible cut), we obtain the inclosure contradiction at $X = \Omega$:

$$\delta(\Omega) \notin \Omega \text{ and } \delta(\Omega) \in \Omega$$

Inclosure: paradox or limitation theorem?

The Inclosure Schema is a *conditional* result. Two standard readings.

Paradox reading: retain the strong assumptions (unrestricted totality, robust semantic/comprehension principles, etc.). The contradiction is *real*.

Limitation reading: treat the argument as a reductio. Conclude that *at least one* of the generating assumptions fails:

- full *Existence* of Ω,
- *Admissibility* $\theta(\Omega)$,
- availability/definability of the relevant *(diagonal) operator* δ,
- the underlying *closure* principles.

Limits of the axiomatic ideal

A unifying meta-lesson (1931-1936):

- **Gödel:** sufficiently rich axiomatic theories of arithmetic are incomplete.
- **Tarski:** truth for arithmetic is not definable *within* arithmetic.
- **Turing:** the Halting problem is undecidable (no total decision procedure).
- **Church:** first-order validity is undecidable.

These results are paradigm cases of diagonal self-reference turning naive “totality/completeness” assumptions into contradiction, and thus into *limitation theorems*.

Gödel I as an inclosure structure

Totality.
$\varphi(\sigma) \equiv \sigma$ is a true arithmetical sentence, $\qquad \Omega = \{\sigma : \varphi(\sigma)\}$.

Admissibility.

$$\theta(X) \;\equiv\; \begin{cases} X \subseteq \Omega, \\ X \text{ is the set of theorems of some r.e. theory } T_X \supseteq Q. \end{cases}$$

(So X is an *effective* and *sound* fragment of arithmetic truth.)

Diagonal. Given $\theta(X)$, let $\delta(X) = G_X$ be a diagonal sentence satisfying:

$$G_X \;\leftrightarrow\; \neg\mathrm{Prov}_{T_X}('G'_X)$$

This is the Gödel sentence *relative to* T_X.

Gödel I as an inclosure structure

Transcendence. If $\theta(X)$ and T_X is consistent, then

$$G_X \notin X \qquad \text{(i.e. } T_X \nvdash G_X\text{)}$$

Closure. If $\theta(X)$ (soundness), then G_X is true, hence

$$G_X \in \Omega.$$

Thus $\delta(X)$ always *escapes* the admissible X while remaining inside the truth-totality Ω.

From the inclosure pattern to Gödel's limitation theorem

The inclosure "limit step" would be:

$$X = \Omega \quad \text{with} \quad \theta(\Omega)$$

But $\theta(\Omega)$ would amount to:

- **Effectivity:** the set of all arithmetical truths is r.e.,
- **Axiomatizability:** there exists a single r.e. theory proving all arithmetical truths,
- **Soundness:** all its theorems are true.

If we assume this "ideal totality", then we obtain the inclosure contradiction:

$$\delta(\Omega) \notin \Omega \text{ and } \delta(\Omega) \in \Omega$$

We block $\neg\theta(\Omega)$. So there is *no* consistent, r.e. theory extending Q whose theorems coincide with all arithmetical truths. Hence every such theory is incomplete.

Outline

1. Canonical paradoxes
2. The Inclosure Schema
3. Responses

What the arguments show

- Many paradoxes are *conditional*:

$$\text{Completeness/closure assumption} \Rightarrow \perp$$

- The key assumptions are often:
  1. unrestricted totalities,
  2. naive truth principles,
  3. global representability/completeness.

Type-theoretic and set-theoretic restriction

Classic strategy:

- Block Existence of the problematic totality Ω.
- E.g. ZFC avoids a set of all sets, a set of all ordinals, etc.

In Priest's terms: deny Clause (1) of Russell's Schema for the set-theoretic cases.

Tarskian hierarchy

Semantic strategy:

- Stratify truth predicates: $\mathsf{T}_0, \mathsf{T}_1, \ldots$
- Disallow a single truth predicate applying to sentences containing it.

In Priest's terms: revise the principles used to establish Closure/Transcendence in semantic cases.

Kripkean fixed-point theories of truth

Another semantic strategy:

- Use partial or grounded truth assignments.
- Seek a least fixed point of a truth operator.

This can be seen as an approach to banishing certain naive contradictions.

Dialetheic approach

Priest's distinctive proposal:

- Keep the core principles leading to the contradiction.
- Reject Explosion by adopting a paraconsistent logic.
- Accept that certain limit objects/sentences are both true and false.